Summing Up All Genus Free Energy
of ABJM Matrix Model

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Abstract

The localization technique allows us to compute the free energy of the $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory dual to type IIA strings on $AdS_4 \times CP^3$ from weak to strong 't Hooft coupling $\lambda = N/k$ at finite N, as demonstrated by Drukker, Mariño, and Putrov. In this note we study further the free energy at large 't Hooft coupling with the aim of testing AdS/CFT at the quantum gravity level and, in particular, sum up all the $1/N$ corrections, apart from the worldsheet instanton contributions. The all genus partition function takes a remarkably simple form – the Airy function, $\mathrm{Ai}\big((\pi k^2/\sqrt{2})^{2/3}\lambda_{\mathrm{ren}}\big)$, with the renormalized 't Hooft coupling λ_{ren}.

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1 Introduction and main results

There has been considerable progress in testing the AdS/CFT conjecture [1] in the large N limit. Through its connection to the spin chain systems, the large N integrability allowed us to successfully check the duality from weak to strong 't Hooft coupling with remarkably high accuracy [2]. In the meantime, there is hardly any test of the AdS/CFT duality at finite N. On the one hand, the $1/N$ corrections are quantum gravity loop corrections and thus very hard to compute on the gravity side. On the other hand, there are indications from the field theory analysis that the integrability breaks down at the non-planar level [3]. So there was virtually no technique to evaluate the $1/N$ corrections at strong 't Hooft coupling.

However, there has been important progress in the localization technique in supersymmetric gauge theories [4], elucidating ealier works [5, 6]. This provides a powerful method to compute the $1/N$ corrections at strong 't Hooft coupling in dual CFT's. In [7] the localization technique was applied to $\mathcal{N} = 6 \ U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory, also known as the ABJM theory, dual to type IIA strings on $AdS_4 \times CP^3$ or M-theory on $AdS_4 \times S^7/Z_k$ [8]. The partition function of the ABJM theory reduces to the eigenvalue integrals and this defines the ABJM matrix model. Subsequently, the free energy of the ABJM matrix model was computed exactly in the large N limit and order by order in the $1/N$ expansion [9, 10, 11]. In particular, the genus zero free energy at strong 't Hooft coupling precisely agrees with the tree level SUGRA action. This provided a remarkable check of the AdS/CFT duality and showed the power of the localization technique.[1]

In this note we study further the free energy of the ABJM theory at strong 't Hooft coupling at finite N with the aim of testing AdS/CFT at the quantum gravity level and, in particular, resum all the $1/N$ corrections. Our main result is that, apart from the worldsheet instanton contributions, the all genus free energy of the ABJM matrix model sums up to

$$F^{\text{ABJM}} = \log \left[2\pi C_1 \text{Ai} \left(\left[\frac{\pi}{\sqrt{2}} \left(\frac{N}{\lambda} \right)^2 \lambda_{\text{ren}}^{3/2} \right]^{2/3} \right) \right] , \qquad (1.1)$$

where $C_1 = (2\pi/k)^{-1/3}/\sqrt{2}$ and $\lambda = N/k$ is the 't Hooft coupling of $\mathcal{N} = 6 \ U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory, and

$$\lambda_{\text{ren}} = \lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2} . \qquad (1.2)$$

This non-planar shift/renormalization of the 't Hooft coupling (1.2) partially resums the all genus free energy [11]. As we will see, once this partial resummation is done, the remaining higher genus free energies obey a very simple recursion relation [16] which one can easily solve to find the main result (1.1).

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[1]See related interesting developments; generalizations to quiver gauge theories [12], the F-maximization conjecture [13], computations of superconformal index [14], and tests of three-dimensional dualities [15].

It is worthwhile to note that the shift (1.2) is closely related to the renormalization of the AdS radius (in the $\alpha' = 1$ unit) [17]:

$$R_{AdS}^2 = 2^{5/2}\pi\sqrt{\lambda - \frac{1}{24} + \frac{\lambda^2}{24N^2}} \ . \tag{1.3}$$

In the planar limit $N \to \infty$, the matrix model shift (1.2) agrees with that of the dual string theory (1.3), as noted in [9, 10].[2] However, there appears to be a discrepancy at the non-planar level; The matrix model suggests a further shift $-3\lambda^2/8N^2$ to the string/M theory prediction. This may provide a simplest test of AdS/CFT at the quantum gravity level, and it is very important to understand whether and how this discrepancy can be reconciled. We will return to this point later in the discussions.

The organization of the rest of our paper is as follows: In section 2 we briefly review the localization technique and the ABJM matrix model and summarize the result of [9] relevant to our study. In section 3 we give a brief review of the technique to compute the higher genus free energies by using the so-called holomorphic anomaly equation and carry out a partial resummation of the all genus free energy. In section 4 we propose the all genus free energy up to the worldsheet instanton corrections and give a proof of our proposal. In section 5 we summarize our results and discuss the SUGRA one-loop corrections to the free energy. In the appendices we give the technical details of our computations.

2   The ABJM matrix model

The localization technique [4] allows us to compute the free energy of the ABJM theory from weak to strong 't Hooft coupling $\lambda = N/k$ at finite N. The localization of this theory on S^3 was carried out by Kapustin, Willet, and Yaakov in [7]; The partition function reduces to the finite dimensional integral over the eigenvalues of two $U(N)$ matrices. This defines the ABJM matrix model. As demonstrated by Drukker, Mariño, and Putrov [9, 11, 10], the free energy can be calculated for arbitrary 't Hooft coupling λ and order by order in the $1/N$ expansion.

The field content of the ABJM theory consists of two 3d $\mathcal{N} = 2$ $U(N)$ vector multiplets, $(A_\mu^A, \sigma^A, \lambda^A, \bar{\lambda}^A, D^A)_{A=1,2}$, which are the dimensional reduction of 4d $\mathcal{N} = 1$ vector multiplets and two bifundamental chiral multiplets, $(\phi^I, \bar{\phi}^I, \psi^I, \bar{\psi}^I, F^I, \bar{F}^I)_{I=1,2}$, in the representation $(N, \bar{N})$, and their duals in $(\bar{N}, N)$.[3] This theory enjoys the nilpotent Grassmann-odd symmetry generated by $\bar{\delta} = \bar{\epsilon}\bar{Q}$ where $\bar{\epsilon}$ being the two-component complex Killing spinor on S^3 with the normalization $\bar{\epsilon}\epsilon = 1$ and $\bar{Q}$ is the $\mathcal{N} = 1$ supercharge.

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[2]In [9] the unequal rank $U(N_1) \times U(N_2)$ case [18] was also studied, and the generalization of (1.2) precisely agrees with the SUGRA prediction [19] in the large N limit.

[3]In Euclidean space, the barred fields are not complex conjugate of the unbarred fields but independent of them.

The partition function is invariant under the deformation of the action by any $\bar\delta$-exact terms, since the $\mathcal{N} = 6$ $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter action and the vacua are invariant under the $\bar\delta$-transformation. This has an important ramification: By deforming the action by $t\bar\delta V$ with a positive definite $\bar\delta V$, as we send t to infinity, the path integral localizes on the saddle point of $\bar\delta V$. A particularly convenient choice of $\bar\delta V = \bar\delta V_{\text{gauge}} + \bar\delta V_{\text{matter}}$ is [7, 20]

$$\bar\delta V_{\text{gauge}} = \bar\delta\delta\text{Tr}_{U(N)}\left(\frac{1}{2}\bar\lambda^A\lambda^A - 2D^A\sigma^A\right) = S_{\text{YM}} , \tag{2.1}$$

$$\bar\delta V_{\text{matter}} = \bar\delta\delta\left(\bar\psi^I\psi^I - 2i\bar\phi^I\left(\sigma^1 - \sigma^2\right)\phi^I - \bar\phi^I\phi^I\right) = S_{\text{m}} , \tag{2.2}$$

where S_{YM} and S_{m} are the $\mathcal{N} = 2$ super Yang-Mills action and the bifundamental matter action on the unit sphere, respectively. Their saddle points are given by [7]

$$A_\mu^A = \phi^I = 0 , \qquad D^A = -\sigma^A = \text{const} . \tag{2.3}$$

The localization action $S_{\text{YM}} + S_{\text{m}}$ vanishes on the saddle points. Thus, in the $t \to \infty$ limit, the only contribution comes from (1) the classical Chern-Simons-matter action S_{cl} evaluated on the saddle points and (2) the quadratic fluctuations about the saddle points in the localization action. The latter gives the one-loop determinants, and thus schematically the partition function becomes

$$Z^{\text{ABJM}} = \int d\sigma^1 d\sigma^2 \frac{\det \Delta_F(\sigma^A)}{\det \Delta_B(\sigma^A)} \exp\left(-S_{cl}(\sigma^A)\right) , \tag{2.4}$$

where Δ_B and Δ_F denote the Laplacians on S^3 for the bosonic and fermionic fluctuations, respectively.

Diagonalizing the $U(N)$ matrices $\sigma^{i=1,2}$ and integrating their angular parts yields the Vandermonde determinants which are cancelled by the factors from the one-loop determinants. The net result is given by [7, 22, 9, 10]

$$Z^{\text{ABJM}} = \frac{1}{(N!)^2}\int \prod_{i=1}^N \frac{d\mu_i}{2\pi}\prod_{a=1}^N \frac{d\nu_a}{2\pi}\frac{\prod_{i<j}\left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2 \prod_{a<b}\left(2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right)^2}{\prod_{i,a}\left(2\cosh\left(\frac{\mu_i-\nu_a}{2}\right)\right)^2}$$
$$\times e^{-\frac{1}{2g_s}\left(\sum_i \mu_i^2 - \sum_a \nu_a^2\right)} , \tag{2.5}$$

where $g_s = 2\pi i/k$. μ_i's and ν_a's are the eigenvalues of the two $U(N)$ matrices σ_i, and the hyperbolic functions are the one-loop determinant contributions. Those in the numerator are from the vector multiplets, whereas those in the denominator are from the bifundamental matter multiplets which give the coupling between the two $U(N)$ factors. The eigenvalue integral (2.5) defines the ABJM matrix model.

The planar limit of the ABJM matrix model can be solved by the standard technique [9, 11, 10]. In the large N limit, the partition function (2.5) is dominated by the saddle

points:

$$\mu_i = \frac{t_1}{N_1} \sum_{j \neq i}^{N_1} \coth \frac{\mu_i - \mu_j}{2} + \frac{t_2}{N_2} \sum_{a=1}^{N_2} \tanh \frac{\mu_i - \nu_a}{2} \, , \tag{2.6}$$

$$\nu_a = \frac{t_2}{N_2} \sum_{b \neq a}^{N_2} \coth \frac{\nu_a - \nu_b}{2} + \frac{t_1}{N_1} \sum_{i=1}^{N_1} \tanh \frac{\nu_a - \mu_i}{2} \, , \tag{2.7}$$

where $t_i = g_s N_i$ and $N_1 = -N_2 = N$. In solving these equations, one first considers the problem for positive N_1 and N_2 and then analytically continues the result to the negative $N_2 = -N_1 = -N$.

The effective potential which leads to the above saddle point equations consists of attractive harmonic potentials and logarithmic Coulomb repulsions for the eigenvalues μ_i's and $\nu_a + i\pi$'s. Since the Coulomb repulsions have the strength t_i's, when the 't Hooft couplings t_i's are zero, the harmonic potentials dominate and the eigenvalues collapses to zero. As we increase t_i's, the Coulomb repulsions kick in and spread the eigenvalues along two line intervals; The μ_i's condense in $\mathcal{C}_1$ on the real axis, and the $\nu_a + i\pi$'s in $\mathcal{C}_2$, the interval separated from $\mathcal{C}_1$ by $+i\pi$. Thus the ABJM matrix model is a two-cut model.

Similarly to the case of the standard matrix models, one defines the resolvent by

$$\omega(z) = g_s \left\langle \sum_{i=1}^{N_1} \coth \frac{z - \mu_i}{2} \right\rangle + g_s \left\langle \sum_{a=1}^{N_2} \tanh \frac{z - \nu_a}{2} \right\rangle \, . \tag{2.8}$$

To make more direct contact with the standard matrix models, it is useful to introduce the new variable $Z = e^z$. Then the resolvent is expressed as

$$\omega(z) dz = -t \frac{dZ}{Z} + 2g_s \left\langle \sum_{i=1}^{N_1} \frac{dZ}{Z - e^{\mu_i}} \right\rangle + 2g_s \left\langle \sum_{a=1}^{N_2} \frac{dZ}{Z + e^{\nu_a}} \right\rangle \, , \tag{2.9}$$

where $t = t_1 + t_2$. As studied in [21], the saddle point equations (2.6) and (2.7) imply that the discontinuity of the resolvent $\omega_0(z) \equiv \lim_{N \to \infty} \omega(z)$ at large N is given by

$$z = \frac{1}{2} \left(\omega_0(z + i\epsilon) + \omega_0(z - i\epsilon) \right) \quad \text{on} \quad \mathcal{C}_1 \, , \tag{2.10}$$

$$z = \frac{1}{2} \left(\omega_0(z + i\pi + i\epsilon) + \omega_0(z + i\pi - i\epsilon) \right) \quad \text{on} \quad \mathcal{C}_2 \, . \tag{2.11}$$

It then follows that the function

$$f(Z) = e^t \left(e^{\omega_0} + Z^2 e^{-\omega_0} \right) \tag{2.12}$$

is regular everywhere on the complex plane. This has the asymptotic behavior $f(Z) \overset{Z \to \infty}{\longrightarrow} Z^2$ and $f(Z) \overset{Z \to 0}{\longrightarrow} 1$. With these boundary conditions, the function $f(Z)$ is uniquely determined:

$$f(Z) = Z^2 - \zeta Z + 1 \, . \tag{2.13}$$

Meanwhile, from (2.12) one has

$$\omega_0(Z) = \log\left[\frac{e^{-t}}{2}\left(f(Z) - \sqrt{f(Z)^2 - 4e^{2t}Z^2}\right)\right] . \qquad (2.14)$$

Parameterizing the branch cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ on the Z-plane by $[1/a, a]$ and $[-b, -1/b]$, respectively, the inside of the square root can be written as the polynomical $(Z-a)(Z-1/a)(Z+b)(Z+1/b)$, and one can then identify

$$\zeta = \frac{1}{2}\left(a + \frac{1}{a} - b - \frac{1}{b}\right) , \qquad e^t = \frac{1}{4}\left(a + \frac{1}{a} + b + \frac{1}{b}\right) . \qquad (2.15)$$

Once the resolvent is found, the 't Hooft couplings t_i can be computed from

$$t_i = \frac{1}{4\pi i}\oint_{A_i} \omega_0(z)dz , \qquad (2.16)$$

and the genus zero free energy from

$$\frac{\partial F_0}{\partial s} = \pi i t - \frac{1}{2}\oint_B \omega_0(z)dz , \qquad (2.17)$$

where $s = \frac{1}{2}(t_1 - t_2)$. The contour A_i encircles the branch cut $\mathcal{C}_i$, whereas the contour B is the cycle dual to the A_i cycles. The first term in the RHS of (2.17) is due to the pole at $Z = 0$ in (2.9). These contour integrals (2.16) and (2.17) are hard to carry out. However, once the derivative is taken w.r.t. ζ, it becomes easy to perform the integration.

After the analytic continuation $t_2 \to -t_1 = -2\pi i\frac{N}{k} = 2\pi i\lambda$, the answer for the ABJM matrix model turns out to be [9, 11, 10]

$$\lambda(\kappa) = \frac{\kappa}{8\pi}{}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16}\right) , \qquad (2.18)$$

$$\partial_\lambda F_0(\kappa) = \frac{\kappa}{4}G_{3,3}^{2,3}\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, -\frac{1}{2} \end{array}\middle| -\frac{\kappa^2}{16}\right) + \frac{\pi^2 i\kappa}{2}{}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16}\right) \qquad (2.19)$$

where $\kappa = -i\zeta$. In particular, at strong coupling $\lambda \gg 1$, these yield

$$\lambda - \frac{1}{24} = \frac{\log^2\kappa}{2\pi^2} + \mathcal{O}(1/\kappa^2) , \qquad (2.20)$$

$$F_0(\lambda) = \frac{4\sqrt{2}\pi^2}{3}\left(\lambda - \frac{1}{24}\right)^{3/2} + \mathcal{O}\left(e^{-2\pi\sqrt{\lambda - \frac{1}{24}}}\right) . \qquad (2.21)$$

Quite remarkably, the first equation shows that, at least in the large N limit, the free energy of the ABJM theory at strong coupling is naturally expressed in terms of the shifted 't Hooft coupling $\lambda - \frac{1}{24}$, rather than λ, in agreement with the SUGRA prediction [17]. Moreover, the first term in the genus zero free energy precisely agrees with the (minus of) classical

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SUGRA action. As $\lambda = N/k$, this in particular reproduces the tantalizing $N^{3/2}$ scaling of the M2-brane theory. Lastly, the exponential corrections to the free energy agree with the expected worldsheet instanton corrections.

These results proclaim the power and relevance of the localization technique to the test of the AdS/CFT conjecture. With this remarkable success, it is natural to ask if the AdS/CFT can be tested beyond large N limit by studying the non-planar corrections in the ABJM matrix model.

3 Higher genus free energies

The ABJM matrix model is equivalent to the so-called Lens space matrix model [23, 24], which computes the partition function of the $U(N_1) \times U(N_2)$ branch of the Chern-Simons theory on the Lens space $L(2,1) = S^3/Z_2$, by the analytic continuation $N_2 \to -N_2$ [22].

Meanwhile, as discussed in [24], the Lens space matrix model arises as the low energy effective theory of D-branes wrapping S^3/Z_2 in the topological A-model on $T^*(S^3/Z_2)$. At large N the geometric transition takes place and S^3/Z_2 is replaced by the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, yielding the topological A-model on a non-compact Calabi-Yau space, the canonical line bundle over $\mathbb{F}_0$. This is the large N (open/closed string) duality between the Lens space matrix model and the topological A model on local $\mathbb{F}_0$. Furthermore, the A-model on local $\mathbb{F}_0$ can be mapped to the B-model on the mirror manifold given by the surface $uv = H(x,y)$, where $H(x,y) = 0$ is an elliptic curve and coincides with the spectral curve of the Lens space matrix model. The closed string amplitudes $F^{(g)}(t,\bar{t})$ of the B-model, where t is the complex structure moduli, obey the holomorphic anomaly equation of [25] and can be explicitly calculated. In [26] it was conjectured that the holomorphic limit $F^{(g)}(t) \equiv \lim_{\bar{t}\to\infty} F^{(g)}(t,\bar{t})$ of the amplitudes are given by the genus g free energies of the matrix model whose spectral curve is $H(x,y) = 0$. Direct proof was given in [27] built on [28] that the solution of the matrix model loop equation [29] is that of the holomorphic anomaly equation in the $\bar{t} \to \infty$ limit when the appropriate boundary conditions are imposed. Thus one can use the holomorphic anomaly equation to find the genus g free energies of the ABJM matrix model which is equivalent to the Lens space matrix model by a simple analytic continuation.

3.1 The holomorphic anomaly equation

Parameterizing the complex structure moduli by the coordinates t^I, the holomorphic anomaly equation takes the form [25]: $(g \geq 2)$

$$\partial_{\bar{I}} F_g = \frac{1}{2} C_{\bar{I}\bar{J}\bar{K}} e^{2K} G^{J\bar{J}} G^{K\bar{K}} \left(D_J D_K F_{g-1} + \sum_{r=1}^{g-1} D_J F_r D_K F_{g-r} \right) , \qquad (3.1)$$

where $G_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$ is the Kähler metric on the moduli space, C_{IJK} is the Yukawa coupling defined by $C_{IJK} = \partial_I \partial_J \partial_K F_0$ with F_0 being the genus 0 free energy, and the covariant derivative D_I acts on F_g and its derivatives as $\partial_I - \Gamma^\bullet_{I\bullet} + (2 - 2g)\partial_I K$. In the local Calabi-Yau case, the Kähler potential is given by $K = \frac{i}{2}\left(t^K \partial_{\bar{K}} \bar{F}_0 - t^{\bar{K}} \partial_K F_0\right)$, and thus the moduli space metric is $G_{I\bar{J}} = \mathrm{Im}\tau_{IJ}$ with $\tau_{IJ} \equiv \partial_I \partial_J F_0$ [16].

When applied to the ABJM matrix model [9], the moduli space becomes one-dimensional and the modulus t is identified with the 't Hooft coupling λ. Then the Yukawa coupling is $C_{\lambda\lambda\lambda} = 4\partial_\lambda^3 F_0(\lambda) = -32\pi^3 i\xi$ where

$$\xi = \frac{2}{\vartheta_2(\tau)^2 \vartheta_4(\tau)^4} , \qquad \tau = \frac{i}{4\pi^3}\partial_\lambda^2 F_0(\lambda) + 1 = i\frac{K'\left(\frac{i\kappa}{4}\right)}{K\left(\frac{i\kappa}{4}\right)} \tag{3.2}$$

with the normalizations properly adjusted. The genus g free energies are assumed to be of the form

$$F_g(\tau) = \xi^{2g-2} f_g(\tau) , \tag{3.3}$$

$$f_g(\tau) = \sum_{k=1}^{3g-3} \frac{A_g^{(k)}(\tau)}{k}\left(\frac{E_2(\tau)}{12}\right)^k + c_g^0(\tau) , \tag{3.4}$$

where $A_g^{(k)}(\tau)$'s are the modular forms of weight $6g - 6 - 2k$, and $c_g^0(\tau)$ is the holomorphic ambiguity and a modular form of weight $6g - 6$. The Eisenstein series of weight 2, $E_2(\tau)$, is a quasi-modular form and can be promoted to the non-holomorphic modular form as

$$E_2(\tau) \longrightarrow \hat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\mathrm{Im}\tau} . \tag{3.5}$$

By promoting $F_g(\tau)$ to the non-holomorphic modular form $F_g(\tau, \bar{\tau})$ in this way, the non-holomorphicity comes in only through E_2. It then follows from the holomorphic anomaly (3.1) that ($g \geq 2$)

$$\frac{df_g}{dE_2} = -\frac{1}{3}\left\{d_\xi^2 f_{g-1} + \frac{1}{3}\frac{\partial_\tau \xi}{\xi}d_\xi f_{g-1} + \sum_{r=1}^{g-1} d_\xi f_r d_\xi f_{g-r}\right\} , \tag{3.6}$$

where the covariant derivative d_ξ transforms a form of weight k to a form of weight $k+2$: $d_\xi = \partial_\tau + \frac{k}{3}\frac{\partial_\tau \xi}{\xi}$. In Appendix A we collect various formulae used in the subsequent computations.

The genus one free energy can be independently calculated by using Akemann's formula [30]. One can find that

$$F_1(\tau) = -\log \eta(\tau) , \tag{3.7}$$

where $\eta(\tau)$ is the Dedekind eta function. This is the initial data needed to solve the holomorphic anomaly equation. In particular, at strong coupling $\lambda \sim \log^2 \kappa \gg 1$, this is expanded as

$$F_1(\tau) = \frac{1}{6}\log \kappa - \frac{1}{2}\log\left(\log \kappa\right) + \mathcal{O}(1/\kappa) . \tag{3.8}$$

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3.2 The weight zero free energy

We first focus on $A_g := A_g^{(3g-3)}$'s of modular weight zero. By using the formulae in Appendix A, it is easy to show that they obey a simple recursion relation [16]

$$\frac{A_g}{12} = -\frac{1}{3}\left((3g-3)A_{g-1} + \sum_{r=2}^{g-2} A_r A_{g-r}\right) . \tag{3.9}$$

They are the highest order terms in F_g's in the $x = 1/\log\kappa$ expansion. Indeed, using the modular transformation properties in Appendix A, one finds that

$$F_g = \frac{(-1)^{g-1}}{4^{2g-2}}\frac{A_g}{3g-3}x^{3g-3} + \mathcal{O}\left(x^{3g-4}\right) := F_g^{[0]}x^{3g-3} + \mathcal{O}\left(x^{3g-4}\right) , \tag{3.10}$$

where $x = 1/\log\kappa$. As it will turn out, the leading order recursion (3.9) carries almost all the information needed to sum up the all genus free energy, when the worldsheet instanton corrections are ignored.

We can in fact resum the highest order all genus free energy

$$F(t) = \sum_{g=2}^{\infty} F_g^{[0]} t^{2g-2} , \tag{3.11}$$

where $t = -ig_s x^{3/2}$. For the convenience, we instead consider the generating function

$$f(u) := \sum_{g=2}^{\infty} \tilde{A}_g u^g , \tag{3.12}$$

where $\tilde{A}_g = (g-1)(4/3)^{g-1}F_g^{[0]}$. Then it is easy to find that

$$f(u) - \tilde{A}_2 u^2 = u^2 f(u)' + f(u)^2 \tag{3.13}$$

with $\tilde{A}_2 = 5/36$ which can be found from the initial genus one data. This can be solved to

$$f(u) = -\frac{d}{du^{-1}}\log H(u) , \tag{3.14}$$

where

$$H(u) = e^{-\frac{1}{2u}}\sqrt{1/2u}\left(C_1 K_{1/3}(1/2u) + C_2 I_{1/3}(1/2u)\right) , \tag{3.15}$$

with C_1 and C_2 being integration constants. The highest order all genus free energy can then be computed as

$$F(t) = -\int^{4/3t^2} du^{-1} f(u) = \log H(3t^2/4) . \tag{3.16}$$

Note that we absorbed the integration constant into C_1 and C_2.

To fix the constants C_1 and C_2, we require the absence of the non-perturbative corrections of the type $e^{-2/3t^2}$ for small t. Since $t \sim g_s$, this is of the order e^{-1/g_s^2} and would be the gravitational instanton or NS5-brane effect, as opposed to the D-brane effect, in the dual AdS string theory. We assume that the type IIA string theory on $AdS_4 \times CP^3$ does not receive such corrections. With this assumption, there remain two choices; (1) $C_2 = 0$ or (2) $C_1 = -(-1)^{5/6}C_2/\pi$. However, the latter choice is subject to a further non-perturbative ambiguity due to the Stokes phenomenon of the modified Bessel function $I_{1/3}(z)$. So we claim that the correct choice is $C_2 = 0$. The remaining constant C_1 only shifts the constant part of the genus one free energy and is not physically important.

Using the identity $\mathrm{Ai}(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}}K_{1/3}\left(\frac{2}{3}z^{3/2}\right)$, we find

$$F(t) = \log\left[2\pi C_1 e^{-\frac{2}{3t^2}} t^{-\frac{1}{3}} \mathrm{Ai}(t^{-4/3})\right] , \tag{3.17}$$

where $t = -ig_s x^{3/2}$. Note that the factors in front of the Airy function are precisely the minus of the highest order terms (in x) of the genus zero and one contributions. Hence the total weight zero free energy becomes simply

$$\widehat{F}^{[0]}(g_s, x) = \log\left[2\pi C_1 \mathrm{Ai}\left(\left(-g_s^2 x^3\right)^{-2/3}\right)\right] . \tag{3.18}$$

As we will see momentarily, we have actually resummed a significant part of the all genus free energy up to the worldsheet instanton corrections.

4   The all genus free energy

It was observed in [11] that the all genus free energy, apart from the worldsheet instanton corrections, depends on the 't Hooft coupling only through the shifted/renormalized form

$$\lambda_{\mathrm{ren}} = \lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2} . \tag{4.1}$$

This translates to the renormalization of $g_s x$ by

$$g_s x \to g_s y \equiv \frac{g_s x}{\sqrt{1 + (g_s x)^2/6}} , \tag{4.2}$$

where $g_s = 2\pi i \lambda/N$ and $x^{-1} = \sqrt{2}\pi\sqrt{\lambda - 1/24}$.

We now claim that, apart from the worldsheet instanton corrections, the all genus free energy can be obtained by replacing x with the renormalized variable y in the weight zero free energy (3.18). That is, the all genus free energy of the ABJM theory is given by

$$F^{\mathrm{ABJM}}(\lambda, N) = \log\left[2\pi C_1 \mathrm{Ai}\left(\left(-g_s^2 y^3\right)^{-2/3}\right)\right] + \mathcal{O}\left(e^{-2\pi\sqrt{\lambda - \frac{1}{24}}}\right) , \tag{4.3}$$

where $C_1 = (-g_s^2)^{-1/6}/\sqrt{2}$ to match the normalization in [11]. This is the main result (1.1).

To show it, we first note that, when the worldsheet instanton corrections, *i.e.*, the $\mathcal{O}(1/\kappa)$ terms, are neglected, the holomorphic anomaly equation (3.6) becomes

$$F_g'(x) = \frac{1}{4} x^4 F_{g-1}''(x) + \frac{12x-1}{12} x^2 F_{g-1}'(x) + \frac{x^4}{4} \sum_{r=2}^{g-2} F_r'(x) F_{g-r}'(x) \ . \tag{4.4}$$

To derive this, we performed the modular transformation $ST^{-1} : \tau \mapsto \tau' = -1/(\tau - 1)$, and kept only the terms in powers of $\log \kappa = \pi i \tau'/2 + \mathcal{O}\left(e^{2\pi i \tau'}\right)$. We then used the formulae listed in Appendix A.

To proceed, we define the generating function

$$\mathcal{F}(g_s, x) := \sum_{g=2}^{\infty} g_s^{2g-2} F_g'(x) \tag{4.5}$$

which obeys

$$\mathcal{F}(g_s, x) - g_s^2 F_2'(x) = \frac{1}{4} g_s^2 x^4 \partial_x \mathcal{F}(g_s, x) + \frac{12x-1}{12} g_s^2 x^2 \mathcal{F}(g_s, x) + \frac{1}{4} g_s^2 x^4 \mathcal{F}(g_s, x)^2 \ , \tag{4.6}$$

where

$$F_2(x) = \frac{x}{144} - \frac{x^2}{24} + \frac{5x^3}{48} \ . \tag{4.7}$$

Then the claim (4.3) is equivalent to

$$\mathcal{F}(g_s, x) = \frac{\partial y}{\partial x} \left\{ \frac{4}{g_s^2 y^4} \widehat{f}\left(\frac{3}{4} g_s^2 y^3\right) - \partial_y \left[\frac{2}{3 g_s^2 x^3} + \left(\frac{1}{6x} + \frac{1}{2} \log x\right) \right] \right\} \ , \tag{4.8}$$

where

$$\widehat{f}(z) := f(z) + \left[-\frac{1}{2} + \frac{z}{6} \right] \ . \tag{4.9}$$

is the generating function (3.14) plus the genus zero and one contributions, while the second term in (4.8) is the subtraction of the genus zero and one contributions. Plugging (4.8) into (4.6) yields

$$f\left(\frac{3}{4} g_s^2 y^3\right) - \frac{5}{36}\left(\frac{3}{4} g_s^2 y^3\right)^2 = \left(\frac{3}{4} g_s^2 y^3\right)^2 f'\left(\frac{3}{4} g_s^2 y^3\right) + f\left(\frac{3}{4} g_s^2 y^3\right)^2 \ . \tag{4.10}$$

This is equivalent to (3.13). Hence we have shown that the all genus free energy (4.3) indeed satisfies the holomorphic anomaly equation (3.6) up to the worldsheet instanton corrections.

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5 Discussions and conclusions

We succeeded to sum up the all genus free energy of the ABJM theory, apart from the worldsheet instanton corrections. The resummation was done in two steps; (1) First, we summed up the modular weight zero part of the all genus free energy as given in (3.18). (2) Then the remaining parts were resummed to (4.3) by renormalizing the 't Hooft coupling as was done in [11]. The renormalization (1.2) agrees with the SUGRA prediction [17] in the large N limit. However, there seems to be a discrepancy at the non-planar level; The matrix model suggests a further shift $-3\lambda^2/8N^2$ to the string/M-theory prediction (1.3).

The quantum gravity one-loop shift $+\lambda^2/24N^2$ of the AdS radius in (1.3) accounts for, at the least, a part of the one-loop corrections. This comes from the shift of the M2/D2-brane charge [17]

$$N \to N - \frac{1}{24}\left(k - \frac{1}{k}\right) = k\left(\lambda - \frac{1}{24} + \frac{\lambda^2}{24N^2}\right) , \tag{5.1}$$

due to the higher curvature correction $C_3 \wedge I_8$, where I_8 is a curvature 8-form anomaly polynomial [31]. One might wonder if the one-loop discrepancy $-3\lambda^2/8N^2$ comes from other SUGRA one-loop corrections such as the R^4 term [32].[4] However, we now argue that they are absent: In fact, the one-loop R^4 term is nonzero and of the order $\sqrt{\lambda}$ which agrees with the leading part of the genus one free energy in (3.8). However, the numerical factor does not agree with the $-3\lambda^2/8N^2$ discrepancy, and the R^4 term is not the only one-loop correction. The one-loop R^4 term is expected to be completed by the terms involving the 4-form F_4. Indeed, the 11d $\mathcal{N} = 1$ SUGRA on-shell superfield [33] suggests schematically the completion of the form $(R + F_4^2)^4$, but this vanishes on $AdS_4 \times S^7/Z_k$ [34, 35]. Furthermore, the one-loop vacuum energy vanishes in 4d $\mathcal{N} \geq 5$ SUGRA on the global AdS_4 whose boundary is $\mathbb{R} \times S^2$ [36]. Although we work in the Euclidean AdS_4 with the boundary S^3, the localizations of the ABJM theory on S^3 and $\mathbb{R} \times S^2$ turn out to be the same [37]. So we expect the SUGRA result on the global AdS_4 applies to our case, and it seems likely that the one-loop corrections, except for the charge shift (5.1), are absent.

This leaves us the $-3\lambda^2/8N^2$ discrepancy and suggests us to look for the resolution on the matrix model side. Recall that ABJM proposed two $\mathcal{N} = 6$ superconformal Chern-Simons-Matter theories [8]; One has the gauge group $U(N) \times U(N)$ and the other has $SU(N) \times SU(N)$. In the large N limit, their difference may not matter. But at finite N they will differ from each other in the $1/N$ corrections. In fact, in the $AdS_5 \times S^5$ case, the correct gauge group is $SU(N)$ rather than $U(N)$ [38]. Although the status of this subtlety is unclear in the AdS_4 case, it may be worthwhile to investigate $SU(N) \times SU(N)$ ABJM matrix model to see if the $-3\lambda^2/8N^2$ discrepancy can be resolved. We hope to address this issue further in the near future.

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[4]The λ^2/N^2 shift appears of the order $\sqrt{\lambda}$ in the genus one free energy.

Finally, it is worth emphasizing the remarkable simplicity of our result; The partition function of the ABJM theory, when the worldsheet instantons are neglected, is simply the Airy function. This might suggest a possible connection of the ABJM theory to the Kontsevich matrix model [39] upon the inclusion of the worldsheet instanton corrections. Meanwhile, in a somewhat different context of M-theory flux compactification, it was argued that the norm square of the "wave-function of the universe" or the 5d black hole entropy was given by the Airy function [40].[5] It would be interesting to study the relations to these works in the future.

Acknowledgment

A   Formulae for holomorphic anomaly equation

In this appendix, we gather various useful formulae related to the holomorphic anomaly equation:

$$b \equiv \vartheta_2(\tau)^4 , \quad c \equiv \vartheta_3(\tau)^4 , \quad d \equiv \vartheta_4(\tau)^4 ,$$

$$\frac{\partial_\tau \xi}{\xi} = \frac{b - E_2}{4} , \quad d_\xi f_1 = -\frac{E_2}{24} , \quad d_\xi b = \frac{1}{3}b(b+d) , \quad d_\xi d = -\frac{1}{6}d(b+d) ,$$

$$d_\xi E_2 = \frac{2bE_2 - E_2^2 - E_4}{12} , \quad d_\xi E_4 = \frac{bE_4 - E_6}{3} , \quad d_\xi E_6 = \frac{bE_6 - E_4^2}{2} . \tag{A.1}$$

The following formulae are useful to derive the weight zero recursion (3.9)

$$f_g = \frac{A_g^{(3g-3)}}{3g-3}\left(\frac{E_2}{12}\right)^{3g-3} + \cdots ,$$

$$d_\xi f_g = -A_g^{(3g-3)}\left(\frac{E_2}{12}\right)^{3g-2} + \cdots ,$$

$$d_\xi^2 f_g = (3g-2)A_g^{(3g-3)}\left(\frac{E_2}{12}\right)^{3g-1} + \cdots . \tag{A.2}$$

To study the perturbative terms as in (4.4), we shall use the modular transformations ST^{-1} : $\tau \mapsto \tau' = -1/(\tau - 1)$, since (3.2) implies that τ is given by $\tau = 1 + \pi i/(2\log\kappa)$, neglecting

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[5]The authors are grateful to Hirosi Ooguri for pointing this out to us. See also [41].

the $\mathcal{O}(1/\kappa)$ instanton terms,

$$
\begin{aligned}
b(\tau) &= -(-i\tau')^2 d(\tau') = (\tau')^2[1 + \mathcal{O}(1/\kappa)] \;, \\
d(\tau) &= (-i\tau')^2 c(\tau') = (\tau')^2[-1 + \mathcal{O}(1/\kappa)] \;, \\
E_2(\tau) &= (\tau')^2[E_2(\tau') - 6i/(\pi\tau')] = (\tau')^2[(1 - 3x) + \mathcal{O}(1/\kappa)] \;, \\
\xi^2(\tau) &= 4/\big(b(\tau)d(\tau)^2\big) = (\tau')^{-6}[4 + \mathcal{O}(1/\kappa)] \;,
\end{aligned}
\tag{A.3}
$$

with $x = 1/\log\kappa$. Note that since the holomorphic anomaly equation is a modular covariant equation, the modular scale factors τ' are destined to be cancelled out in the end. We shall drop them in our following analysis. Then, we obtain the following formulae in the approximation neglecting the $\mathcal{O}(1/\kappa)$ terms:

$$
d_\xi E_2 = -\frac{3}{4}x^2 \;, \quad d_\xi^2 E_2 = -\frac{3}{8}x^3 \;, \quad \frac{1}{3}\frac{\partial_\tau \xi}{\xi} = \frac{x}{4} \;, \quad \frac{dE_2}{dx} = -3 \;, \quad d_\xi b = d_\xi d = 0 \;. \tag{A.4}
$$

From these we find

$$
\begin{aligned}
\frac{df_g}{dE_2} &= -\frac{1}{3}f_g'(x) \;, \\
d_\xi f_g &= (d_\xi E_2)\frac{df_g}{dE_2} = \frac{x^2}{4}f_g'(x) \;, \\
d_\xi^2 f_g &= (d_\xi^2 E_2)\frac{df_g}{dE_2} + (d_\xi E_2)^2\frac{d^2 f_g}{dE_2^2} = \frac{x^3}{8}f_g'(x) + \frac{x^4}{16}f_g''(x) \;.
\end{aligned}
\tag{A.5}
$$

B   A derivation of weight zero free energy

In this appendix we solve the differential equation (3.13): Introducing the new variable $v = 1/(2u)$ and the new function

$$
f = -\frac{1}{2}\frac{d}{dv}\log H \;, \tag{B.1}
$$

this becomes

$$
\left[\frac{d^2}{dv^2} + 2\frac{d}{dv} + \frac{\tilde{A}_2}{v^2}\right]H = 0 \;. \tag{B.2}
$$

In terms of $G(v) = e^v v^{-\frac{1}{2}}H(v)$, one obtains ($\alpha = 1/3$)

$$
\left[v^2\frac{d}{dv} + v\frac{d}{dv} - (v^2 + \alpha^2)\right]G = 0 \;, \tag{B.3}
$$

which is nothing but the modified Bessel's differential equation. The solution is given by the modified Bessel's functions: $G(v) = C_1 K_{1/3}(v) + C_2 I_{1/3}(v)$.

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